1 Derivation of wavelet transformation normalisation factor

Consider a finite time series, expressed continuously as a function of time:

\[ x = x(t) \quad | \quad t \in [0, T] \]  \hspace{1cm} (1)

We define a Morlet wavelet as

\[ \phi(a,t) = b \exp \left( -i\omega_0 t \frac{t}{a} \right) \exp \left( -\frac{t^2}{2a^2} \right). \] \hspace{1cm} (2)

The choice of the normalisation parameter \( b \) depends on the application. Typically we want to normalise the individual wavelet and choose \( b = \pi^{-0.25}a^{-0.5} \). In this case we get

\[ \phi(a,t) = \pi^{-0.25}a^{-0.5} \exp \left( -i\omega_0 t \frac{t}{a} \right) \exp \left( -\frac{t^2}{2a^2} \right), \] \hspace{1cm} (3)

\[ \int_{t=-\infty}^{\infty} \phi(a,t)\phi^*(a,t) = \pi^{-0.5}a^{-1} \int_{t=-\infty}^{\infty} \exp \left( -\frac{t^2}{a^2} \right). \] \hspace{1cm} (4)

If we define \( \sigma = a/\sqrt{2} \) we can use the normalisation of the Gaussian curve:

\[ \int_{t=-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp \left( -\frac{t^2}{2\sigma^2} \right) = 1 \] \hspace{1cm} (5)

\[ \int_{t=-\infty}^{\infty} \exp \left( -\frac{t^2}{2\sigma^2} \right) = \sigma\sqrt{2\pi} \] \hspace{1cm} (6)

and obtain

\[ \int_{t=-\infty}^{\infty} \phi(a,t) = \pi^{-0.5}a^{-1}\sigma\sqrt{2\pi} = 1. \] \hspace{1cm} (7)

Let’s Fourier transform the wavelet. To do so, we Fourier-transform parts of the formula first:

\[ \exp \left( -\frac{t^2}{2a^2} \right) = \int_{\omega=-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} a\sqrt{2\pi} e^{-\omega^2 a^2}, \] \hspace{1cm} (8)

\[ \exp \left( -i\omega_0 \frac{t}{a} \right) = \int_{\omega=-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} 2\pi \delta(\omega - \omega_0 \frac{1}{a}), \] \hspace{1cm} (9)

\[ \int_{t=-\infty}^{\infty} \phi(a,t) = \int_{\omega=-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} 2\pi \delta(\omega - \omega_0 \frac{1}{a}), \] \hspace{1cm} (10)
Since $\phi(a, t)$ is a product, we can write $\phi(a, \omega)$ as a convolution:

$$\phi(a, t) = \pi^{-0.25} a^{-0.5} \int_{\omega = -\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \left( a \sqrt{2\pi e^{-\frac{\omega^2}{2}}} \right) \ast \left( 2\pi \delta(\omega - \frac{\omega_0}{a}) \right)$$ (11)

$$= \pi^{0.25} \sqrt{2a} \int_{\omega = -\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \left( e^{-\frac{(a\omega)^2}{2}} \right) \ast \left( \delta(\omega - \frac{\omega_0}{a}) \right)$$ (12)

$$= \pi^{0.25} \sqrt{2a} \int_{\omega = -\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} e^{-\frac{(a\omega - \omega_0)^2}{2}}.$$ (13)

So we obtain the Fourier transform as

$$\hat{\phi}(a, \omega) = \pi^{0.25} \sqrt{2a} e^{-\frac{(a\omega - \omega_0)^2}{2}}.$$ (14)

In practice, we want to calculate a convolution of the wavelet and the time series:

$$W(a, t) = \int_{\tau = -\infty}^{\infty} d\tau x(\tau) \phi(a, t - \tau).$$ (15)

Since this is a convolution, we can write it as a product in Fourier space:

$$W(a, t) = \int_{\omega = -\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{x}(\omega) \tilde{\phi}(a, \omega).$$ (16)

In practice, we do the wavelet transform in the following way: We start from a time series $x_1, \ldots, x_N$ which is the function $x(t)$ sampled with an equidistant time step $\Delta t$. It is padded with zeros (between $N/\sqrt{2}$ and $N\sqrt{2}$ such that the resulting length is a power of 2). Then $\tilde{x}_k$ is the fast Fourier transform of this padded time series.

Next we define a set of temporal scales $a$. For each of these $a$, we calculate the product in Fourier space as

$$\tilde{W}(a, \omega) = \tilde{x}(\omega) \pi^{0.25} \sqrt{2a} e^{-\frac{(a\omega - \omega_0)^2}{2}}.$$ (17)

Back-transformation then gives the wavelet amplitude $W(a, t)$.

Let’s take a look at the units which we get. Assume for simplicity that $x$ has the unit m. Time is measured in seconds, so $a$ has the unit s and $\omega$ has the unit s$^{-1}$. The relative scale $\omega_0$ is dimensionless.

Equation (3) states that $\phi(a, t)$ has the unit s$^{-0.5}$. So from equation (15) we can derive a unit for $W(a, t)$ which is m s$^{0.5}$ (note that $dt$ has a unit of s). This is very impractical since this unit has no direct physical meaning.

Let’s consider alternative normalisations. For example, let $x(t)$ be a harmonic oscillation with a period matching that of the Wavelet and an amplitude of 1 m:

$$x(t) = (1 \text{ m}) \exp \left( -i \frac{\omega_0}{a} t \right).$$ (18)
We may want the wavelet to have an amplitude of \( x_0 = 1 \) m as well. In this case, we obtain:

\[
x_0 = |W(a, t)| = \left| \int_{\tau = -\infty}^{\infty} d\tau \; x(\tau) \phi(a, t - \tau) \right| \tag{19}
\]

\[
= \left| \int_{\tau = -\infty}^{\infty} d\tau \; x_0 \exp\left( -i\frac{\omega_0}{a}\tau \right) b \exp\left( -i\frac{\omega_0}{a} t - \tau \right) \exp\left( -\frac{(t - \tau)^2}{2a^2} \right) \right| \tag{20}
\]

\[
= \left| x_0 b \exp\left( -i\frac{\omega_0}{a} t \right) \right| \left| \int_{\tau = -\infty}^{\infty} d\tau \exp\left( -\frac{(t - \tau)^2}{2a^2} \right) \right| \tag{21}
\]

\[
b = a^{-1}(2\pi)^{-0.5} \tag{22}
\]

However, in reality, we only have the real part of the signal. Since both the real and the imaginary part will contribute the same to the amplitude, we need to multiply by a factor of two. Yes, it is a factor of two and not of \( \sqrt{2} \) because we never square the signal \( x(t) \) so \( W(a, t) \) depends linearly on \( x(t) \). We end up with

\[
b = a^{-1}\pi^{-0.5}\sqrt{2} \tag{23}
\]

This means the wavelet is defined as

\[
\phi(a, t) = \frac{\sqrt{2}}{\sqrt{\pi a}} \exp\left( -i\frac{\omega_0}{a} t \right) \exp\left( -\frac{t^2}{2a^2} \right) \tag{24}
\]

If we compare to the previous value of

\[
b_{old} = \pi^{-0.25} a^{-0.5} \tag{25}
\]

we find that we need to correct by a factor of \( \pi^{-0.25} a^{-0.5} \sqrt{2} \) and obtain

\[
\tilde{W}(a, \omega) = \tilde{x}(\omega)2e^{-\frac{(a\omega - \omega_0)^2}{2}} \tag{26}
\]